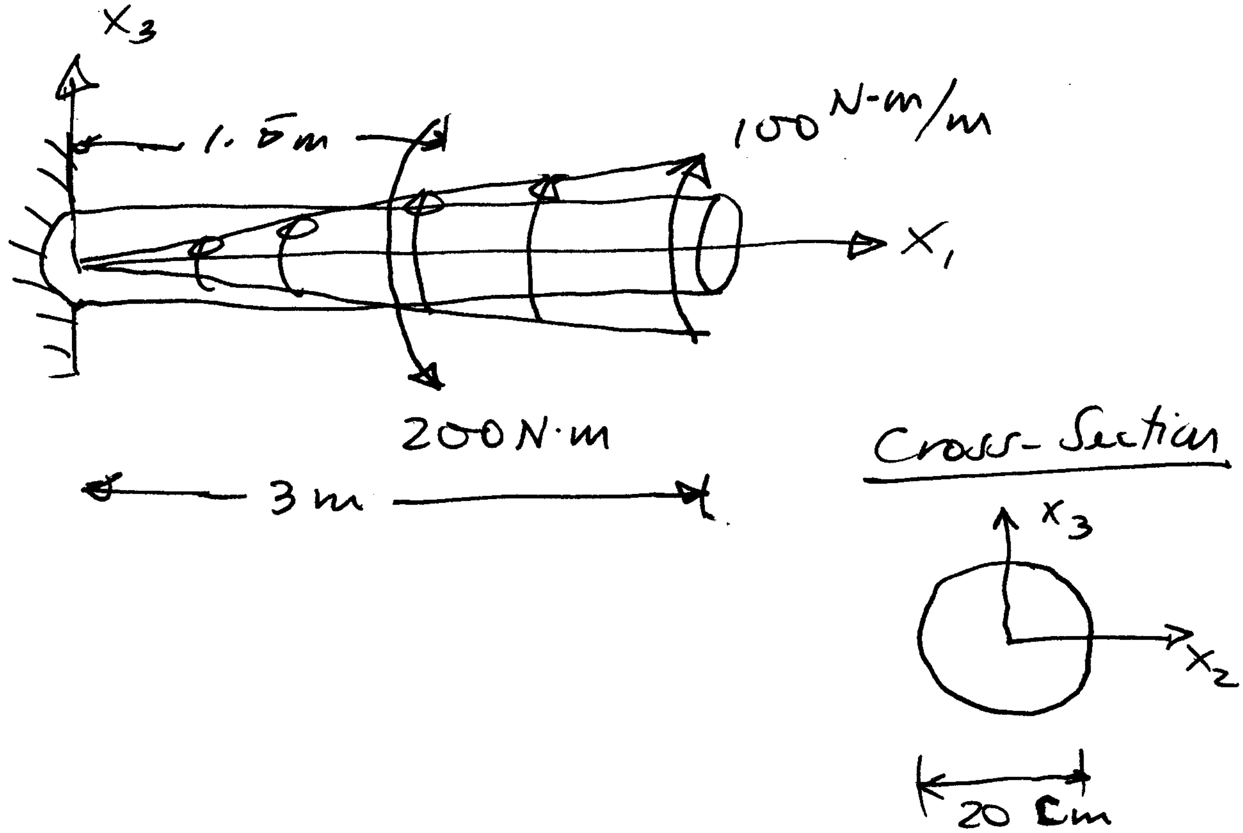
PAL
3/15/08

Unified Engineering Problem Set
Week 7 Spring, 2008

SOLUTIONS

M 7.1

Aluminum rod



(a) Begin by determining equation for the distributed torque.

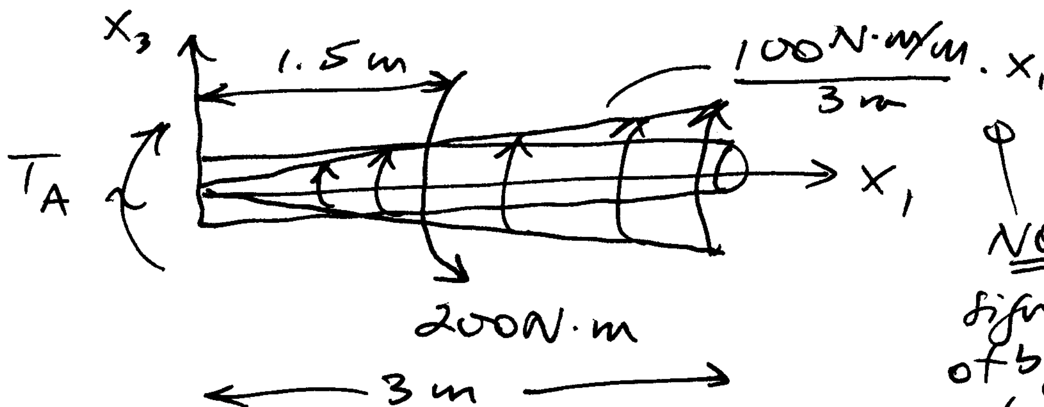
- it is negative
- it equals 0 at $x_1 = 0$
- it has intensity of 100 N·m/m at $x_1 = 3\text{m}$

This results in

$$t(x_1) = -\frac{100\ N\cdot m/m}{3m} \cdot x_1 = -33.3\ \frac{N\cdot m}{m^2}\, x_1$$

(NOTE: units are still torque intensity ⇒ correct)

→ Now draw Free Body Diagram



NOTE: Negative sign taken care of by direction of torque

Take equilibrium of moments/torques about x_1. Use right hand rule (RHR) for +:

$$\sum T_{x_1} = 0 \ \overset{RHR}{\underset{+}{\curvearrowright}} x_1 \Rightarrow -T_A + 200\ N\cdot m - \int_0^{3m} 33.3\ \frac{N\cdot m}{m^2}\, x_1\, dx_1 = 0$$

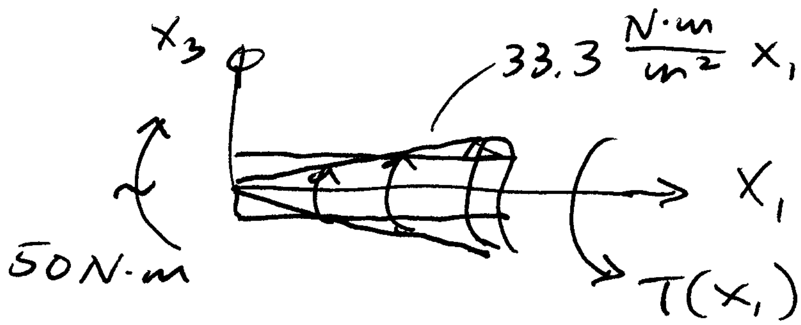
$$\Rightarrow T_A = 200\ N\cdot m - 33.3\ \frac{N\cdot m}{m^2}\left(\frac{x_1^2}{2}\right)\Bigg]_0^{3m}$$

$$= 200\ N\cdot m - 150\ N\cdot m$$

Resulting in: $T_A = 50\ N\cdot m$

→ To determine the torque distribution, $T(x_1)$, make cuts in the shaft/rod in the regions of different loading.

<u>First region</u> $0 < x_1 < 1.5\ m$



Take equilibrium:

$$\sum T_{x_1} = 0 \quad \overset{RHR}{\underset{+}{\curvearrowright}} \Rightarrow -50\ N\cdot m - \int_0^{x_1} 33.3 \frac{N\cdot m}{m^2} x_1' dx_1' + T(x_1) = 0$$

$$\Rightarrow T(x_1) = 50\ N\cdot m + 33.3 \frac{N\cdot m}{m^2}\left(\frac{x_1'^2}{2}\right)\Bigg]_0^{x_1}$$

giving:

$$\boxed{T(x_1) = 50\ N\cdot m + 16.7 \frac{N\cdot m}{m^2} x_1^2}$$

$0 < x_1 < 1.5\ m$

<u>OR</u> use $\frac{dT}{dx_1} = -t(x_1)$

$$= 33.3 \frac{N\cdot m}{m^2} x_1$$

$$\Rightarrow T(x_1) = \int 33.3 \frac{N\cdot m}{m^2} x_1 \, dx_1$$

$$= 33.3 \frac{N\cdot m}{m^2} \left(\frac{x_1^2}{2}\right) + c_1$$

Use Boundary Condition that at $x_1 = 0$, T opposes the reaction torque $\Rightarrow T(0) = 50\ N\cdot m$

this gives $C_1 = 50\ N\cdot m$

$$\Rightarrow T(x_1) = 16.7 \frac{N\cdot m}{m^2} x_1^2 + 50\ N\cdot m$$

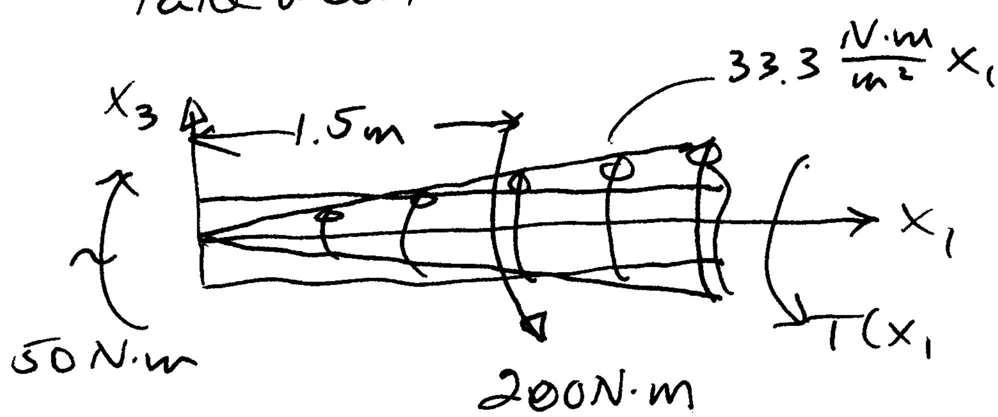
$0 < x_1 < 1.5\ m$

same result (as it must be)

Proceed to ....

Second region $\quad 1.5\ m < x_1 < 3\ m$

take a cut ....



Take equilibrium:

$$\sum T_{x_1} = 0 \quad \overset{RHR}{\circlearrowleft}_{+} \Rightarrow -50\ N\cdot m + 200\ N\cdot m - \int_0^{x_1} 33.3 \frac{N\cdot m}{m^2} x_1' \, dx_1' + T(x_1) = 0$$

$$\Rightarrow T(x_1) = -150\ N\cdot m + 33.3\,\frac{N\cdot m}{m^2}\left(\frac{x_1'^2}{2}\right)\Big]_0^{x_1}$$

giving:

$$\boxed{T(x_1) = -150\ N\cdot m + 16.7\,\frac{N\cdot m}{m^2}\,x_1^2}$$

$$1.5\,m < x_1 < 3\,m$$

<u>OR</u> use $\frac{dT}{dx_1} = -t(x_1)$

$$= 33.3\,\frac{N\cdot m}{m^2}\,x_1$$

again giving: $T(x_1) = 33.3\,\frac{N\cdot m}{m^2}\left(\frac{x_1^2}{2}\right) + C_2$

Now need a boundary condition within this region. Could use the interface with region 1 or use the easier condition at the free end ($x_1 = 3\,m$) that there is no torque resultant: $T(3\,m) = 0$

$$\Rightarrow\quad T(3\,m) = 0 = 150\ N\cdot m + C_2$$

$$\Rightarrow C_2 = -150\ N\cdot m$$

This results in:

$$T(x_1) = 16.7\,\frac{N\cdot m}{m^2}\,x_1^2 - 150\ N\cdot m$$

$$1.5\,m < x_1 < 3\,m$$

again <u>same</u> <u>result</u>

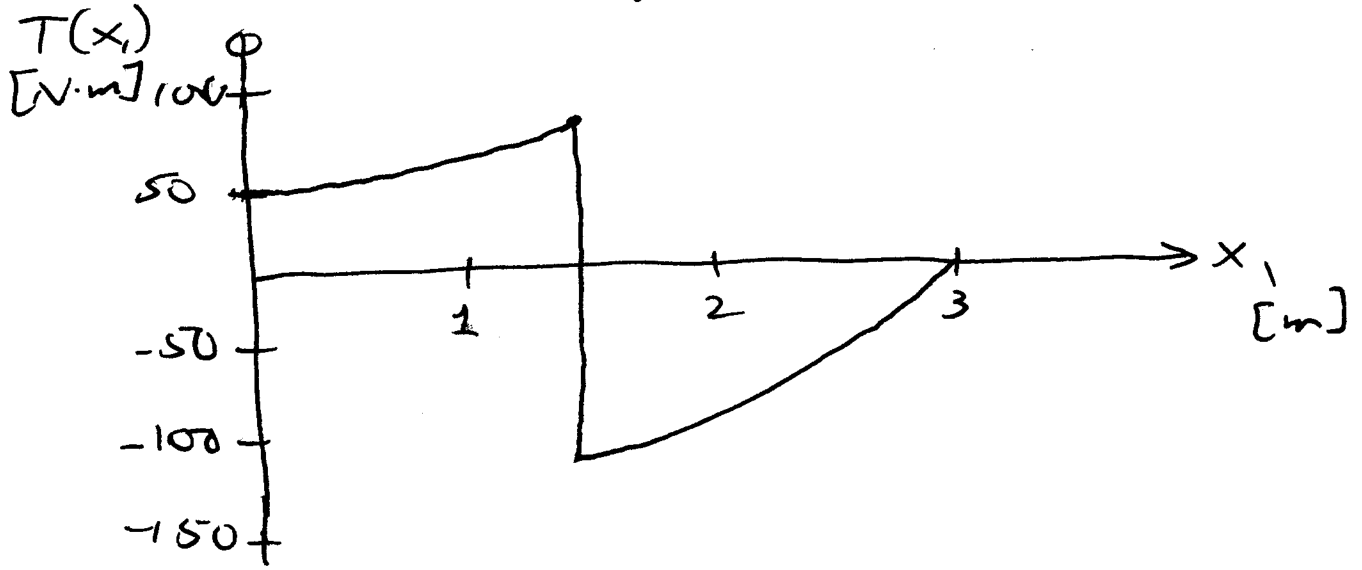
Now sketch the result - - - .

Note key points:

- @ $x_1 = 0$, $T(x_1) = 50\ N\cdot m$
- @ $x_1 = 1.5^- m$, $T(x_1) = 87.5\ N\cdot m$
- @ $x_1 = 1.5^+ m$, $T(x_1) = -112.5\ N\cdot m$

  (NOTE: Jump of $-200\ N\cdot m$ ... equal and opposite to applied point-torque)
- @ $x_1 = 3\ m$, $T(x_1) = 0$

and $T(x_1)$ varies quadratic and positive in the two segments:



(b) To determine the twist of the rod, use the equation for the twist angle:

$$\frac{d\phi}{dx_1} = \frac{T}{GJ}$$

Consider each region starting at the point of the boundary condition on the twist angle: $\phi(x_1 = 0) = 0$

So for <u>Region 1</u>:

$$\phi = \frac{1}{GJ}\int\left(50\,N\cdot m + 16.7\,\frac{N\cdot m}{m^2}\,x_1^{\,2}\right)dx_1$$

$$\Rightarrow \phi(x_1) = \frac{1}{GJ}\left(50\,N\cdot m\,x_1 + \frac{16.7}{3}\,\frac{N\cdot m}{m^2}\,x_1^{\,3}\right) + C_3$$

$$0 < x_1 < 1.5\,m$$

using the Boundary condition gives:

$$C_3 = 0$$

So:

$$\phi(x_1) = \frac{1}{GJ}\left(50\,N\cdot m\,x_1 + 5.56\,\frac{N\cdot m}{m^2}\,x_1^{\,3}\right)$$

$$0 < x_1 < 1.5\,m$$

Proceed to <u>Region 2</u> .....:

again using: $\dfrac{d\phi}{dx_1} = \dfrac{T}{GJ}$

with $T(x_1)$ for this region:

$$\Rightarrow \frac{d\phi}{dx_1} = \frac{1}{GJ}\left(-150\,N{\cdot}m + 16.7\,\frac{N{\cdot}m}{m^2}\,x_1^2\right)$$

so: $$\phi = \frac{1}{GJ}\int\left(-150\,N{\cdot}m + 16.7\,\frac{N{\cdot}m}{m^2}\,x_1^2\right)dx_1$$

giving: $$\phi(x_1) = \frac{1}{GJ}\left(-150\,N{\cdot}m\,x_1 + \frac{16.7}{3}\,\frac{N{\cdot}m}{m^2}\,x_1^3\right) + C_4$$

Get a value to determine C_4 by matching the twist angle at the interface (boundary) with Region 1. From that result:

$$\phi(1.5\,m) = \frac{1}{GJ}\left(93.8\,N{\cdot}m^2\right)$$

matching gives:

$$\phi(1.5\,m) = \frac{1}{GJ}\left(93.8\,N{\cdot}m^2\right) = \frac{1}{GJ}\left(-206.2\,N{\cdot}m^2\right) + C_4$$

$$\Rightarrow C_4 = \frac{1}{GJ}\left(300\,N{\cdot}m^2\right)$$

so:
$$\phi(x_1) = \frac{1}{GJ}\left(-150\,N{\cdot}m\,x_1 + 5.56\,\frac{N{\cdot}m}{m^2}\,x_1^3 + 300\,N{\cdot}m^2\right)$$

$$1.5\,m < x_1 < 3\,m$$

So at the tip:

$$\phi(3\,m) = \frac{1}{GJ}\left(-450\ N\cdot m^2 + 150\ N\cdot m^2 + 300\ N\cdot m^2\right)$$

$$\Rightarrow \boxed{\phi(3\,m) = 0}$$

Could also determine GJ to get the torsional stiffness of the rod and values for the ~~overall~~ expressions of ϕ.

In general: $J = \iint (x_2^2 + x_3^2)\,dA$

We know that for a circular cross-section:

$$J = \frac{\pi R^4}{2}$$

Here: $R = 10\ cm = 0.1\ m$

$$\Rightarrow J = 1.57 \times 10^{-4}\ m^4$$

Now find the shear modulus. We are given $E = 67\ GPa$, $\nu = 0.3$ for aluminum.

For isotropic materials:

$$G = \frac{E}{2(1+\nu)} \Rightarrow G = \frac{67\ GPa}{2(1+0.3)} = 25.8\ GPa$$

So: $GJ = \left(25.8 \times 10^9 \frac{N}{m^2}\right)\left(1.57 \times 10^{-4}\ m^4\right) = 4.05 \times 10^6\ N\cdot m^2$

(c) To find the shear stress use:

$$\tau_{res} = \frac{Tr}{J}$$

To determine the maximum magnitude find the maximum magnitude of $T(x_1)$ and of r:

→ The value of r is maximized along the outer surface of the rod ($r = 10$ cm)

→ $T(x_1)$ has a maximum magnitude of -112.5 N·m at $x_1 = 1.5$ m

Converting to consistent units:

$$\tau_{res} = \frac{(-112.5\ N \cdot m)(10 \times 10^{-2}\ m)}{J}$$

from (b) calculated $J = .57 \times 10^{-4}\ m^4$

So:

$$\tau_{res} = \frac{(-112.5\ N \cdot m)(10 \times 10^{-2}\ m)}{1.57 \times 10^{-4}\ m^4}$$

⇒ maximum magnitude is: (and is negative in direction)

$$\boxed{\tau_{res} = 71.7 \times 10^3\ \frac{N}{m^2} = 0.072\,MPa \text{ at } r = 0.1\ m,\ x_1 = 1.5\ m}$$

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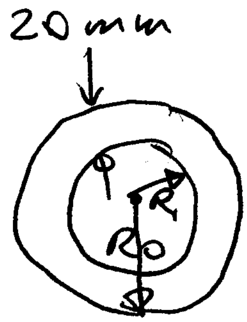
(d) Rod is now a hollow tube with the same outer radius and a wall thickness of 20 mm:



$R_o = 10$ cm

$R_i = 10\text{ cm} - 20\text{ mm}$

$= 8$ cm

The only thing that changes from the solid rod case is the cross-sectional polar moment of inertia (shear modulus stays the same. So ...

→ (a) <u>Torque distribution</u> $\boxed{\text{does not change}}$

This configuration is statically determinate and does not depend upon the cross-sectional properties.

→ (b) All remains the same in considering the twist <u>except</u> J changes.

Can calculate the new J via superposition (remove inner section):

$$J = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2} = \frac{\pi}{2}(R_o^4 - R_i^4)$$

$$= \frac{\pi}{2}(1^4 - 0.8^4)\ [m^4]\ (\times 10^{-4})$$

$$\Rightarrow J = 0.927 \times 10^{-4}\ m^4$$

This gives change for ϕ in general

$$\phi_{new}/\phi_{old} = J_{old}/J_{new} \quad \text{since } \phi \propto \frac{1}{J}$$

$$= \frac{1.57 \times 10^{-4} m^4}{0.927 \times 10^{-4} m^4}$$

$$\Rightarrow \frac{\phi_{new}}{\phi_{old}} = 1.69$$

$\Rightarrow$ ϕ generally increases by 69%

However $\phi_{tip} = 0$ so that value $\boxed{\text{does not change}}$

→ (c) The shear stress is $\tau_{res} = \frac{Tr}{J}$

So only the polar moment of inertia, J, changes as T_{max} stays the same (via (a)) and the maximum value of r does not change.

So the change is via the inverse of J. As shown in part (b)

$$\boxed{\begin{array}{l}\text{maximum magnitude} \\ \quad \tau_{res} \text{ increases by } 69\%\end{array}}$$

location stays at
$r = 0.1\,m$,
$x_1 = 1.5\,m$